Coexisting stochastic and coherence resonance in a mean-field dynamo model for Earth's magnetic field reversals

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Abstract. - Using a spherical symmetric mean field α^2-dynamo model for Earth's magnetic field reversals, we show the coexistence of the noise-induced phenomena coherence resonance and stochastic resonance. Stochastic resonance has been recently invoked to explain the 100 kyr periodicity in the distribution of the residence time between reversals. The comparison of the resulting residence time distribution with the paleomagnetic one allows for some estimate of the effective diffusion time of the Earth's core which may be 100 kyr or slightly below rather than 200 kyr as it would result from the molecular resistivity.

There is ample paleomagnetic evidence that the Earth's magnetic field has undergone polarity changes many times [1]. Averaged over the last few million years the mean rate of reversals is approximately 4-5 per Myr, although the last reversal occurred approximately 780000 years ago. At least two so-called superchrons have been identified as periods of some tens of millions of years with no reversal at all. One of the most intriguing features of reversals is their pronounced asymmetry with the initial decay of the dipole being much slower than the subsequent recreation of the dipole with opposite polarity [2]. Despite the general irregularity of the reversal time series, there are (at least) two features pointing to some sort of ordering. The first one is the clustering property of reversals [3, 4], the second one is the appearance of a 100 kyr periodicity in the distribution of the residence times between reversals [5].

Computer simulations of the geodynamo in general, and of reversals in particular [6], have matured much since the first fully coupled 3D simulation of a reversal by Glatzmaier and Roberts in 1995 [7]. However, the severe problem that dynamo simulations have to work in parameter regions which are far away from the real values of the Earth's core, will remain for a long time. Hence it is certainly useful to understand the essential features of reversals in the framework of simpler models.

With view on the recent successes of liquid sodium dynamo experiments [8, 9] simple reversal models may also help in tailoring future dynamo facilities to make them prone to reversals. As a matter of fact, reversals were observed recently [10] in a special version of the French VKS dynamo experiment that was dominated by the use of iron propellers with a high magnetic permeability [11]. However, the exponential field decay in the initial phase of the experimentally observed reversals seems more indicative for a intermittent extinguishing of the dynamo which is certainly not equivalent to the behaviour of the geodynamo during a reversal. Another interesting dynamo related experiment showing reversals has been reported by Bourgoin et al. [12].

Roughly speaking, there are two classes of simplified dynamo models which try to explain reversals. The first one relies on the assumption that the dominant dipole field is somehow "rotated" from a given orientation to the reversed one via some intermediate state (or states) [13–15]. A typical feature of such models, in particular of the model of Hoyng and coworkers [13,14], is that they work with an effective conductivity of the Earth's core which is strongly decreased compared to the molecular value. The necessity to introduce such a "turbulent conductivity" is well known for the solar dynamo [16], but a dramatic conductivity reduction seems hardly justified for the Earth's core [17]. Another drawback of the "rotation model" is that it results in reversals with the wrong asymmetry as it was shown recently [18]. In spite of these problems, the "rotation model" with a turbulent diffusion time of 5 kyr (instead of 200 kyr as it would result from the molecular resistivity) was successful in recovering the above mentioned 100 kyr periodicity in the distribution of the residence times

between reversals [19]. The underlying physical mechanism was identified as the stochastic resonance (SR) [20], which is an example of noise-induced phenomena in nonlinear systems driven by weak periodic forcing. For the geodynamo, the weak periodic forcing is actually assumed to result from the Milankovich cycle of the Earth's orbit eccentricity [21, 22], although details of the driving mechanism are hard to quantify from first principles.

The second class of simplified reversal models, which could be coined "oscillation models", relies on a spectral peculiarity of the (in general) non-selfadjoint dynamo operator. The basic idea, the specific interplay between a non-oscillatory and an oscillatory branch of the dominant axial dipole mode, was expressed early by Yoshimura [23] and later exemplified within a spherically symmetric mean-field dynamo model of the α^2-type [18, 24–26]. Many features of reversals, in particular the correct time-scale, the mentioned asymmetry and clustering property were attributed to the magnetic field dynamics in the vicinity of a branching point (or *exceptional point* [27]) of the spectrum of the dynamo operator. Usually, this exceptional point, where two real eigenvalues coalesce and continue as a complex conjugated pair of eigenvalues, is associated with a nearby local maximum of the growth rate situated at a slightly lower magnetic Reynolds number. It is the negative slope of the growth rate curve between this local maximum and the exceptional point that makes stationary dynamos vulnerable to noise. Then, the instantaneous eigenvalue is driven towards the exceptional point and beyond into the oscillatory branch where the sign change happens. In this picture, reversals appear as noise-induced relaxation oscillations.

In [24] the model was also shown to exhibit coherence resonance (CR) [28], which is similar to SR but relies on the existence of an intrinsic frequency of the system and not on an external periodic forcing. Given the capability of this model to account for a number of observed reversal features, we will ask in the present paper how the mechanism of SR can be implemented and what we can learn from its coexistence with CR. The coexistence of both resonance types has been demonstrated for the Fitz-Hugh-Nagumo model with the probability density function of the residence times showing a transition from SR to CR for an increasing noise strength [29].

After delineating our simplified mean-field dynamo model (more details and discussions can be found in [18, 24–26]) we will investigate the coexistence of SR and CR. Based on some exploration of the space of governing parameters, we will end up with a conjecture on the effective diffusive time scale of the core. Actually, this effective diffusion time is not well known for two reasons. The first one is that the conductivity σ of the liquid Fe-Ni-Si alloy at the pressure of the Earth's core is hard to determine. The most recent estimate [30] is $\sigma = 4.71 \times 10^5$ $(\Omega \text{ m})^{-1}$. For the Earth's core of radius $R = 3480$ km this would amount to a magnetic diffusion time $T_d := \mu_0 \sigma R^2 = 227$ kyr. The second reason is the already mentioned possibil-

ity that the "molecular" conductivity of the material could be reduced due to the turbulent flow. This so-called β effect [31] is very hard to estimate. Actually, there have been claims [32] on the measurement of a few percent β effect in a turbulent liquid sodium flow with magnetic Reynolds numbers Rm up to 8, but neither the Riga nor the Karlsruhe dynamo experiments have shown evidence of any significant β effect at much larger Rm [8]. However, this negative result might not exclude some β effect in the geodynamo which works at still higher Rm and with a higher degree of turbulence. Although a very strong reduction (by a factor 10 or more) is safely excluded by geomagnetic data, a reduction of σ by a factor 2 or so is not completely forbidden and will be a matter of consideration in this paper.

Our starting point is the well known induction equation [31] for evolution of the magnetic field $\boldsymbol{B}$ under the influence of a helical turbulence parameter α:

$$\frac{\partial \boldsymbol{B}}{\partial \tau} = \boldsymbol{\nabla} \times (\alpha \boldsymbol{B}) + \frac{1}{\mu_0 \sigma} \Delta \boldsymbol{B} \ . \tag{1}$$

After decomposing the magnetic field into a poloidal and a toroidal parts according to

$$\boldsymbol{B} = -\nabla \times (\boldsymbol{r} \times \nabla S) - \boldsymbol{r} \times \nabla T \ , \tag{2}$$

the two defining scalars S and T are expanded in spherical harmonics of degree l and order m. Under the assumption that α is spherically symmetric (which is certainly a grave simplification that does not apply to the Earth's outer core) we arrive, for each degree l and order m separately, at the following pair of partial differential equations:

$$\frac{\partial s_l}{\partial \tau} = \frac{1}{r} \frac{\partial^2}{\partial r^2}(r s_l) - \frac{l(l+1)}{r^2} s_l + \alpha(r, \tau) t_l \ , \tag{3}$$

$$\frac{\partial t_l}{\partial \tau} = \frac{1}{r} \frac{\partial}{\partial r} \left[\frac{\partial}{\partial r}(r t_l) - \alpha(r, \tau) \frac{\partial}{\partial r}(r s_l) \right] \ .$$
$$- \frac{l(l+1)}{r^2}[t_l - \alpha(r, \tau) s_l] \ . \tag{4}$$

Note that in the non-dimensionalized equation system (3-4) the radius r is measured in units of R, the time τ in units of the diffusion time T_d, and α in units of $(\mu_0 \sigma R)^{-1}$. Note further that the order m of the spherical harmonics does not show up in equations (3-4) due to the presupposed spherical symmetry of the problem. The boundary conditions are: $\partial s_l / \partial r|_{r=1} + (l+1) s_l(1) = t_l(1) = 0$. In the following we will consider only the dipole field with $l = 1$.

Beyond a critical intensity of α, the equation system (3-4) will acquire an exponentially growing solution. In reality, however, the magnetic field cannot grow indefinitely, but will attenuate the source of its own generation. We model this attenuation by "quenching" the kinematic α with the angle averaged magnetic field energy which can be expressed in terms of $s(r)$ and $t(r)$. In addition, we assume that α is also influenced by some noise which might
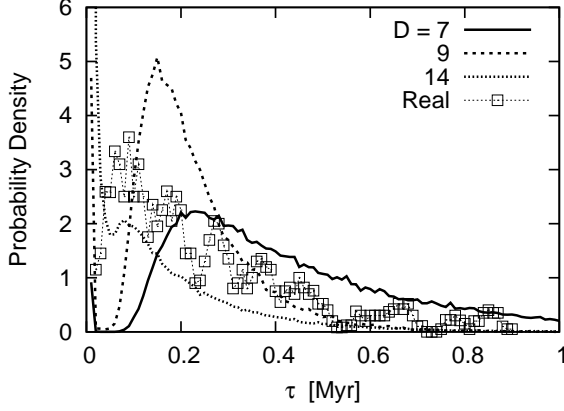
Fig. 1: RTD for $C = 50$, $T_d = 190$ kyr and different values of D in the case without periodic driving, compared to the "real" data from paleomagnetic measurements which were extracted from [5, 19].

Fig. 2: Normalized variance (NV) of the RTD for various C in dependence on D in the case without periodic driving. The minimum at $D \sim 9$ is a typical signature of CR.

be considered as a shorthand for changing boundary conditions or the neglected influence of higher magnetic field modes. Taking into account quenching and multiplicative noise together, we get a time dependent $\alpha(r, \tau)$ in the form

$$\alpha(r, \tau) = \frac{\alpha_{kin}(r)}{1 + E \left[\frac{2s_1^2(r,\tau)}{r^2} + \frac{1}{r^2} \left(\frac{\partial (r s_1(r,\tau))}{\partial r} \right)^2 + t_1^2(r,\tau) \right]}$$
$$+ \xi_1(\tau) + \xi_2(\tau) r^2 + \xi_3(\tau) r^3 + \xi_4(\tau) r^4 , \quad (5)$$

with the noise correlation given by

$$\begin{aligned} < \xi_i(\tau)\xi_j(\tau + \tau_1) > &= D^2 (1 - |\tau_1|/T_c) \\ &\times \Theta(1 - |\tau_1|/T_c)\delta_{ij} , \quad (6) \end{aligned}$$

(Θ is the Heaviside function). In Eqs. (5,6), $\alpha_{kin}(r)$ is the kinematic α profile, D is the noise intensity, E is a constant measuring the inverse mean field energy, and T_c is a correlation time of the noise. As already mentioned the diffusive time scale T_d is approximately 200 kyr if we assume the molecular conductivity of the material in the Earth's core. However, in the following we will consider this value as variable.

Motivated by the earlier observation that α profiles with one sign change along the radius can provide oscillatory solutions [33, 34], we choose for the kinematic α profile in Eq. (5) the Taylor expansion

$$\alpha_{kin}(r) = 1.914 \cdot C \cdot (\alpha_0 + \alpha_1 r + \alpha_2 r^2 + \alpha_3 r^3 + \alpha_4 r^4) \quad (7)$$

with $\alpha_0 = 1$, $\alpha_1 = \alpha_3 = 0$, $\alpha_2 = -6$ and $\alpha_4 = 5$ (the factor 1.914 serves just to normalize C in order to make it comparable to the case of constant α).

For most of the following simulations, the dynamo number is set to $C = 50$ which is highly supercritical compared to the critical value $C_{crit} = 6.8$ which is specific to our particular $\alpha(r)$ profile. The equation system (3-5) is time-stepped by means of a standard Adams-Bashforth
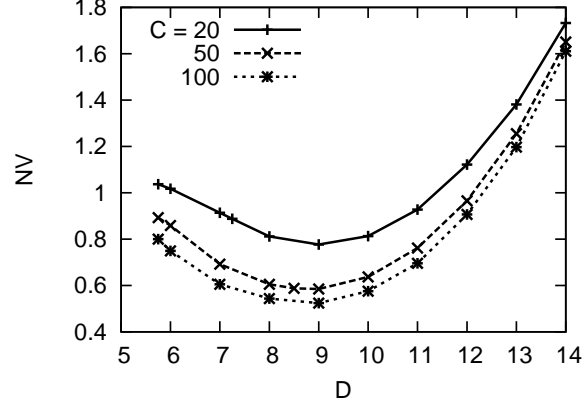
method with radial grid spacing of 0.02 and time step length of 2×10^{-5}. The correlation time T_c has been set to 0.005 which would correspond to 1 kyr in case that the diffusion time is set to 200 kyr. The resulting time series show reversal sequences quite similar to those of the geodynamo [18, 25, 26]. As an important characteristic of these sequences we will determine the distribution of residence times between two subsequent reversals which we will abbreviate by RTD.

We start with the case without any periodic forcing of α. Figure 1 shows the RTD of the dynamo model for different values of the noise intensity D. The diffusion time was set to $T_d = 190$ kyr which is just the double of the 95 kyr period which was already mentioned in [22] and which was also found in [19] as a good fit to the paleomagnetic data. For the sake of comparison, we have included the curve resulting from paleomagnetic measurement as they were published in [5, 19] with its typical SR feature of several maxima around multiples of 95 kyr. All of the numerically resulting curves exhibit a maximum, followed by an exponential decrease of the probability. Starting with $D = 7$, for which the RTD has its maximum at ~ 200 kyr, a trend of this maximum towards smaller values of τ is visible. For $D = 9$ it is located at $\sim$150 kyr and for $D = 14$ it is at $\sim$100 kyr. However, for those larger values of D we observe a much to steep exponential decay, together with an unphysical increase of reversal probability for very small values of τ. This is a first indication that an assumed diffusion time of 190 kyr might be too large to explain the first maximum at ~ 95 kyr observed in the paleomagnetic data.

To illustrate a typical feature of CR we use the normalized variance NV of the residence time distribution, which is defined as follows:

$$NV = \frac{\sqrt{Var(\tau)}}{< \tau >} . \quad (8)$$

Figure 2 shows the dependence of NV on the noise

strength D. For all considered values of C, there is a minimum of NV at $D \sim 9$ which is a clear signature for the occurrence of CR [28]. At this value of the noise intensity we get an optimal amplification of a residence time corresponding to an inherent timescale of the dynamo. The increase of NV for values $D > 9$ is explained by the fact that for large D the dynamo is dominated by the noise and very short residence times become more probable as it was already visible in the curves for $D = 9, 14$ in fig. 1.
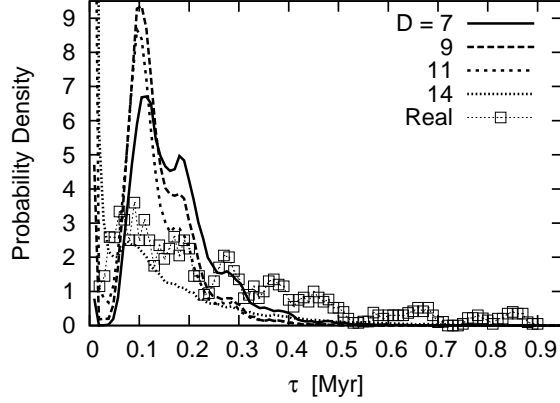
Fig. 3: RTD for $C = 50$, $T_d = 190$ kyr, and different values D in the case with periodic driving with a period of $T_\Omega = 95$ kyr and forcing strength $\epsilon = 0.3$.
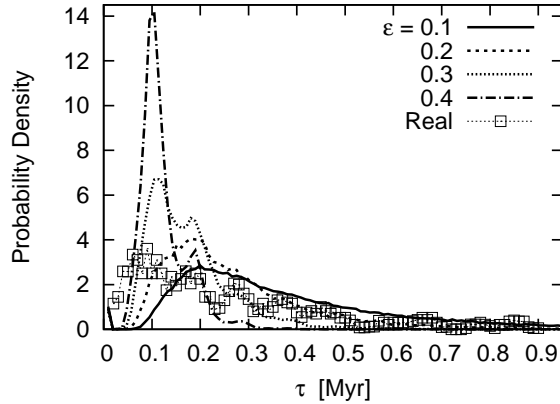

Fig. 4: RTD for $C = 50$, $T_d = 190$ kyr, and different values ϵ in the case with periodic driving with a period of $T_\Omega = 95$ kyr and noise intensity $D = 7$.

After having discussed the pure CR effect, we will now examine the additional occurrence of SR. For this purpose, a weak periodic forcing of α is introduced. Here, "weak" means that there are no reversals in the absence of noise (note that a strong periodic force would dominate the system completely leading to a peak in the residence time distribution located at the timescale of this force). Although both additive and multiplicative noise had been discussed in [19], we focus here only on the effects of multiplicative noise which we consider more physical and

which was also shown to fit better to the paleomagnetic data [19]. Actually, the periodic input is implemented as an additional periodic variation of the helical turbulence parameter α with the period T_Ω. Naively, one would first apply the periodic forcing to the dynamo number C. However, simulations have shown that the dynamo does not react significantly to such a homogeneous periodic input. This insensitivity has to do with the quenching effect which reduces the kinematic α to some quenched α profile which makes the growth rate of the dynamo close to zero. Contrary to such a homogeneous periodic forcing of α, a periodic change of the *shape* of the $\alpha(r)$ profile has a more pronounced effect. As a simplest attempt, we have chosen to add periodic forcing to the first coefficient α_0 of the Taylor expansion of $\alpha_{kin}(r)$ leaving all other coefficients unchanged:

$$\alpha_0(\tau) = 1 + \epsilon \cos(\frac{2\pi}{T_\Omega} \cdot \tau) \qquad (9)$$

For the period T_Ω we chose always the dominant 95 kyr contribution, although the total effect of the Milankovich cycles is certainly much more complicated [22]. The parameter ϵ measures the relative strength of the periodic forcing.
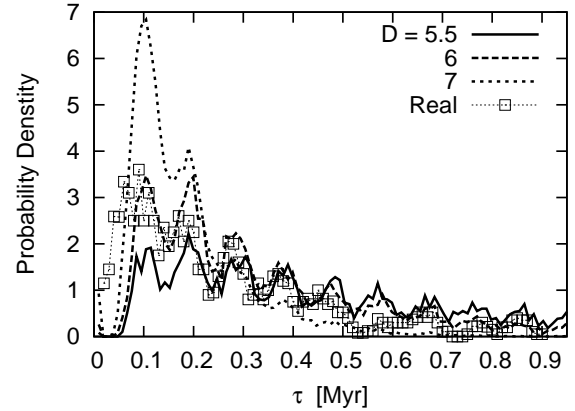

Fig. 5: RTD for $C = 50$, $T_d = 95$ kyr, and different values D in the case with periodic driving with a period of $T_\Omega = 95$ kyr and forcing strength $\epsilon = 0.1$.

The results for $C = 50$, $T_d = 190$ kyr, $T_\Omega = 95$ kyr are shown in fig. 3 where we vary D for fixed $\epsilon = 0.3$, and in fig. 4 where we vary ϵ for fixed $D = 7$. We see in fig. 3, that for D=7 the residence time distribution shows an oscillatory behaviour with a maximum at $\tau = 95$ kyr and further peaks at integer multiples of T_Ω, which decay exponentially. Evidently, the oscillatory behaviour of the probability becomes less pronounced for higher values of D. It is also visible, that for even higher values of D the probability at very small τ increases. The effect of increasing ϵ (for fixed $D = 7$) is quite similar to the effect of increasing D (for fixed ϵ). However, the increase of the probability density for very small τ does not show up here.

Despite some similarity of the curves in figs. 3 and 4 with the paleomagnetic data, with the given $T_\Omega = 95$ kyr
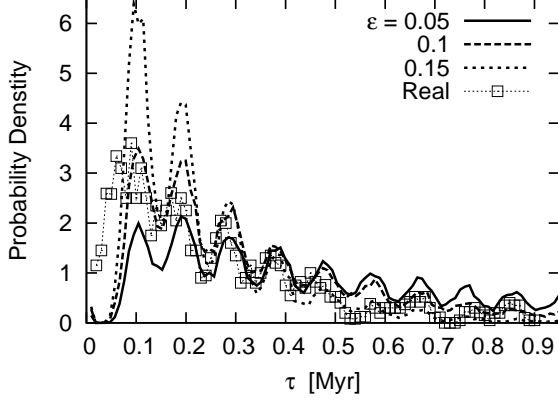
Fig. 6: RTD for $C = 50$, $T_d = 95$ kyr, and different values ϵ in the case with periodic driving with a period of $T_\Omega = 95$ kyr and noise intensity $D = 6$.
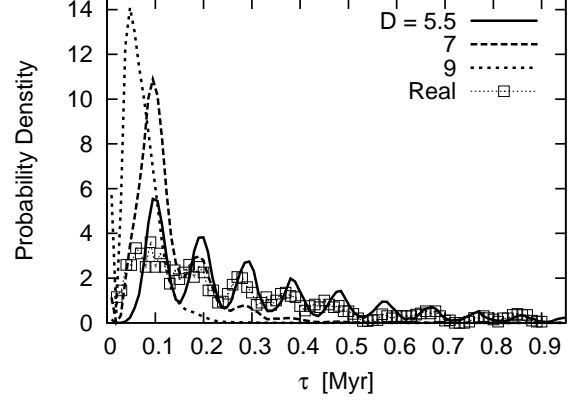

Fig. 7: RTD for $C = 50$, $T_d = 63.3$ kyr and different values D in the case with periodic driving with a period of $T_\Omega = 95$ kyr and forcing strength $\epsilon = 0.1$.

and $T_d = 190$ kyr it seems hard to find parameters ϵ and D which lead to a real good correspondence. This has much to do with the fact that for $T_d > T_\Omega$, which applies to our example, the oscillatory character of the probability distribution starts only at T_d as long as D and ϵ are not too large. This behaviour had already been demonstrated in [29]. To get the first peak at $\tau = T_\Omega = 95$ kyr a rather large value of ϵ and/or D is required which, in turn, leads to a too steep exponential decrease of the probability function.

For this reason, we will test in the following two additional values of the diffusion time, $T_d = 95$ kyr $= T_\Omega$ (figs. 5 and 6) and $T_d = 63.3$ kyr $= 2/3\, T_\Omega$ (figs. 7 and 8). For $T_d = 95$ kyr, $\epsilon = 0.1$, $D = 7$ (fig. 5), but also for $T_d = 95$ kyr, $\epsilon = 0.15$, $D = 6$ (fig. 6) we observe a quite good agreement with the paleomagnetic data, though the ratio between the first and the second peak is not exactly the same.

The version with $T_d = 63.3$ kyr, $\epsilon = 0.1$, $D = 5.5$ (fig. 7) looks also quite promising, although the minima between the probability peaks seem to be more pronounced than in the paleomagnetic data. However, this argument might not be so relevant since in the paleomagnetic data some stronger "smearing out" of the minima could simply result from averaging over a long time period (160 Myr), in which some parameters must undergo some changes in order to explain the long term variation of the reversal rate (cp. [1], p. 184).

Another point is that in the derivation of the histogram of the paleomagnetic data [5], the authors had used a moving box technique which automatically leads to some smoothing of the curve. If we do the same moving box technique to our data, we arrive at the curves shown in fig. 9, which are in reasonable correspondence with the paleomagnetic curve.

To summarize, we have shown that the SR phenomenon which seems to lay at the root of the 95 kyr periodicity of RTD of the Earth's magnetic field reversal can coexist

with the CR phenomenon which is typical for our spherically symmetric α^2 dynamo model. The interesting question is now if the shape of the residence time distribution can serve as a proxy for determining the effective diffusion time of the Earth's core and hence for the reduction of the conductivity due to the β effect. Although we had shown in [18, 26] that a diffusion time of 200 kyr is well compatible with the typical decay and recovery times for individual reversals (as long as some high super-criticality is assumed) we see now that a diffusion time of 100 kyr or a bit below seems more appropriate when it comes to explain the stochastic resonance phenomenon. Finally, this has to do with the effect that the first peak of the stochastic resonance at $\sim$100 kyr is hardly explainable with a diffusion time of $\sim$200 kyr, as long as the strength of the periodic forcing and/or the noise are not forbiddingly large.

Further work will focus on a systematic exploration of the space of parameters C, D, T_d, and ϵ in order to accommodate simultaneously as much paleomagnetic constraints as possible.

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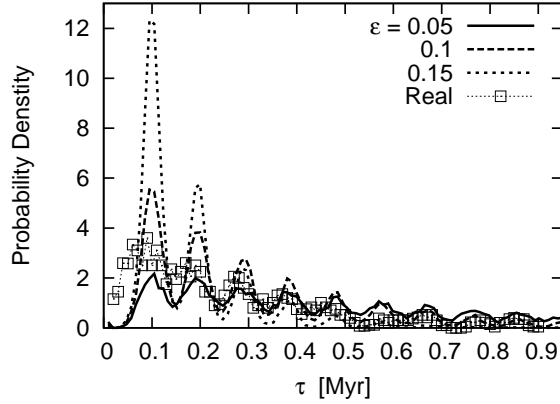[4] Sorriso-Valvo L. et al., *Phys. Earth Planet. Inter.*, **164** (2007) 197.

Fig. 8: RTD for $C = 50$, $T_d = 63.3$ kyr, and different values ϵ in the case with periodic driving with a period of $T_\Omega = 95$ kyr and noise intensity $D = 5.5$.
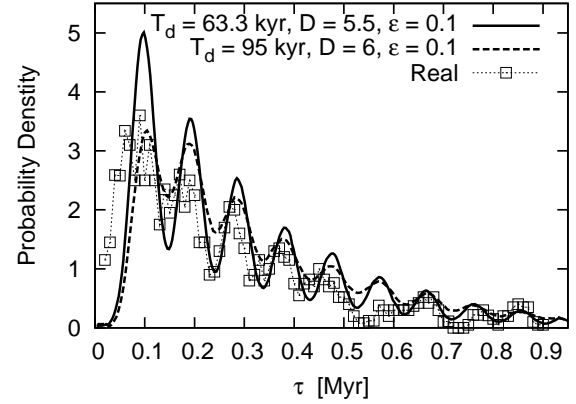

Fig. 9: RTD for the paleomagnetic data and two reasonable parameters for which the moving box technique according to [5] has also been applied.

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